in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

Chapter 5

When Nonrandomness Appears Random: A Challenge to Financial Economics

A. G. Malliaris [†]

[†] *Economics Area, Loyola University Chicago, 25 E. Pearson, Chicago, IL 60611, USA.*

e-mail: tmallia@luc.edu

Mary E. Malliaris[‡]

[‡] *Information Systems Area, Loyola University Chicago, 25 E. Pearson, Chicago, IL 60611, USA.*

e-mail: mmallia@luc.edu

5.1 Introduction

Determinism and randomness are the two pillars of scientific methodology. Ruhla (1992) argues that science, in its long historical evolution, has favored determinism. In other words, the search for an exact relationship between dependent and independent variables has received first priority by scientists who follow the deterministic tradition of Euclid, Newton and Leibnitz. The probabilistic paradigm, which originated in the rigorous analysis of gambling games, has flourished during the past several decades as exact relationships have become more difficult to confirm.

Developments in operations research, management science and financial economics since World War II have reflected the evolution of scientific methodology

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

in the physical sciences. The Marshallian static equilibrium price theory, the Walrasian dynamic tatonement general equilibrium, linear programming techniques, game theory and various other techniques, all emphasized classical determinism. However, measurement errors, unobservable variables, incomplete models, the introduction of expectations and the admission of the economic and business complexity, among other reasons, have swung the methodological pendulum towards probabilistic reasoning. This remarkable shift to probabilistic reasoning is quite evident in financial economics with its key theories of market efficiency and derivatives pricing. The need to forecast an uncertain future variable for purposes of economic and financial planning has reinforced probabilistic methods. Such reasoning gave rise to statistical techniques and the establishment of the field of financial economics. The textbook by Campbell, Lo and MacKinlay (1997) on the econometrics of financial economics exemplifies the probabilistic reasoning in this area.

Although it is currently accepted by economists and financial analysts that there is a clear dichotomy between deterministic and probabilistic modeling, relatively recent developments in physical chaotic dynamics have shown that certain processes, while they appear to be random, need not in fact be random. It is the purpose of this paper to first review rapidly these ideas and, second to consider a model that is deterministic and ask the fundamental question: "When does nonrandomness appear random?". Put differently, suppose that an exact, deterministic theoretical model is developed between certain variables: when or how can a financial economist conclude, by observing exact time series measurements of such variables, that these variables are random?

The remainder of the paper is organized as follows. Section 2 briefly contrasts the notion of deterministic and random models with an emphasis on financial economics, while section 3 presents the most famous deterministic system that behaves like a random one, i.e. the Lorenz equations. Our contribution is exposited in section 4 where we sample from the Lorenz equations and posit the question: when or how can a financial analyst uncover whether the model under analysis is deterministic or random. We illustrate in section 4 that it is possible for a financial economist to conclude that a model is random when actually it is not. An evaluation, summary and questions for further study are given in the last section.

5.2 Deterministic versus random models

Deterministic models consist of exact relationships. Abstracting from specific modelling considerations, the notion of determinism is clearly demonstrated in the relationship of a function:

$$y = f(x) \tag{5.1}$$

where f denotes the set of ordered pairs (x, y). In other words, each x is unambiguously associated with a specific y, with such a y being equal to $f(x)$. From the

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

simple calculus where $f(x) : \mathbb{R} \to \mathbb{R}$, $\mathbb{R}$ denoting the real numbers, to multivariate calculus, differential equations, real analysis and functional analysis, the subject matter remains exact relationships between or among certain variables. These exact relationships can become quite complicated, particularly when such a relationship is between derivatives (i.e. differential equations) or even among functions themselves (i.e. functional analysis). Nevertheless, in all instances, such relationships are exact.

From Euclid's geometry, to Newton's calculus and to today's advanced analysis, the subject matter of scientific investigations is determinism. Discovering, establishing, analyzing and understanding exact relationships among certain variables remains today's highest scientific goal, not only of mathematicians, but also of applied researchers, such as physicists and management scientists. Only after such a primary goal has not been reached, do scientists consider second best solutions by studying nondeterministic models. Such models are also called random or stochastic and are mostly substitutes rather than competing alternatives for the deterministic truth.

Mathematics, which one could argue remains the most rigorous of human scientific efforts, demonstrates that, independent of its intrinsic interest, randomness is not an alternative of equal standing but a temporary substitute to determinism. From the elementary probability, where one flips a coin, to measure-theoretic probability, the notion of a function prevails. What changes is the domain of the function. In probability, the domain is a random set and a function that takes its values from a random set is called a random variable. Ruhla (1992) describes with great scientific care the relationship between these two methodologies by arguing that probability is a branch in the scientific tree of determinism.

The methodological debate between randomness and nonrandomness in financial economics has been extensive. Actually, two very popular books by Malkiel (2003) and Lo and MacKinlay (1999) review extensively the use of random and nonrandom techniques applied to the behavior of stock prices. A more rigorous approach of similar methodological issues is found in Campbell, Lo and MacKinlay (1997), cited in the previous section. Actually, while the efficient market hypothesis celebrated the methodology of random walks and martingales during the 1970s, studies such as Scheinkman and LeBaron (1989) and Hsieh (1989, 1991) followed by Lo (1991), Sengupta and Zheng (1995), Corazza and Malliaris (2002), Kyrtsou and Terraza (2002, 2003) among others, have shown the merit of chaotic dynamics. Useful surveys of the chaotic dynamics methodology can be found in Brock and Malliaris (1989) and Brock, Hsieh and LeBaron (1992).

5.3 The lorenz equations

Our discussion thus far was carried out at the methodological level. In other words, in searching for causal relationships, a scientist may choose an exact or a random

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

model. We have argued that exactness has been given priority in the applied sciences and in pure mathematics, while randomness is viewed as a temporary methodological substitute. How can we further strengthen our argument towards determinism?

Chaotic dynamics was developed precisely for this purpose: to demonstrate that there exist exact functions which generate very complicated trajectories that appear like random. From the seminal work of Eckman and Ruelle (1985) to the numerous texts about dynamics such as Devaney (1986) or Guckenheimer and Holmes (1983), scientists have exposited an exciting new branch of mathematics which reinforces determinism.

Limitations of space do not allow us to describe in detail the key ideas, definitions and theorems of chaotic dynamics. Devaney (1986) presents the essential elements while Guckenheimer and Holmes (1983) treat the subject at a more advanced level. Here, for the sake of continuity, we give the fundamental definition of chaotic dynamics. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is chaotic if it satisfies three conditions: (a) f is topologically transitive, (b) f has sensitive dependence on initial conditions, and (c) f has periodic points that are dense in the real numbers.

May (1976) gives several examples of chaotic maps, while Guckenheimer and Holmes (1983) discuss in detail the mathematical properties of such maps. The Lorenz (1963) equations are the most famous example of a system that generates chaotic dynamics. They are:

$$x_t = s(-x_{t-1} + y_{t-1}) \tag{5.2}$$
$$y_t = r x_{t-1} - y_{t-1} - x_{t-1} z_{t-1} \tag{5.3}$$
$$z_t = -b z_{t-1} + x_{t-1} y_{t-1} \tag{5.4}$$

This system of equations is represented here by difference equations. They can also be expressed as a system of differential equations, as was initially derived by Lorenz (1963) in his meteorological study of a three-equation approximation to the motion of a layer of fluid heated from below. Observe that there are three parameters, s, r and b. More specifically, the parameter r corresponds to the Reynolds number and as it varies, the system goes through remarkable qualitative changes. For parameter values b = 2.667, r = 28.0 and s = 10.0, almost all solutions converge to a set called the strange attractor. Furthermore, once on the attractor, these solutions exhibit random-like behavior. An exhaustive analysis of the numerous properties of these equations may be found in Sparrow (1982). Malliaris (1993) hypothesizes that the S&P 500 Index follows chaotic dynamics and uses neutral networks to confirm this hypothesis. Malliaris and Stein (1999) give a detailed financial interpretation of the Lorenz system and perform an econometric estimation using futures data. Papers by Brock and Hommes (1998), Lux (1998) and Chiarella, Dieci and Gardini (2000) offer theoretical models of chaotic dynamics.

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

5.4 The experiment

We are now in a position to describe our contribution. Using the software Phaser developed by Kocak (1989), we generate 5000 observations using the previous system of equations with parameter values as indicated above. The software generates these values for a choice of two numerical approximation methods, i.e. Euler and Runge-Kutta, and for certain values of the step size in the approximation. Initial values are also needed for the three variables. The values used in our experiment are $x(0) = y(0) = z(0) = 5$.

Notice that for an interval $[0, T]$, Phaser selects a finite number of points $[0, t_1, \ldots, t_k, \ldots, T]$ which for simplicity are chosen to be equally spaced. The distance, $h = t_{k+1} - t_k$, between two consecutive points is called the step size. By selecting a very small step size, let us say 0.01 instead of a larger one, such as 0.1, the numerical approximation becomes more accurate. Of course, such accuracy depends on the particular numerical approximation. Kocak (1989) compares both methods and concludes that the Runge-Kutta approximation is more accurate than the Euler approach. Our calculations are performed using the Runge-Kutta approximation with a step size of 0.1, unless otherwise specified.

The next concept we wish to discuss is the idea of a jump. When for example the jump = 1, solutions are plotted at every step. If the jump = 10, then solutions are plotted at every tenth point. In other words, selecting a jump of 100 means that, although all the necessary calculations are performed, only each hundredth numerical value is sampled. In our experiment, we use jumps of 1, 10, or 100 to check and see whether the techniques used are capable of identifying the deterministic structure of the Lorenz map.

Before we describe our three data sets, and to further motivate our experiment, consider Figures 1 and 2. In Figure 1, we plot the time series of the x-variable for t in $[0, 100]$ when the step size is chosen to be 0.001 and the jump is equal to 1. The strange atttractor is clearly visible and the time series does not appear very random. On the other hand, in Figure 2, for a smaller step size 0.01 and a much larger jump = 100, both the time series of x and its strange attractor lose their structure and appear random-like. Although these two figures do not constitute evidence of randomness, it is instructive to observe that infrequent sampling, by jumping over detailed information, misses the underlying structure of the population data.

Having made the above clarifications, our experiment now can be described. Using the deterministic Lorenz equations, with a step size of 0.1, we generated three sets of 5000 observations each. The first set records each value of the variable x generated by the Lorenz equations. In other words, the first set has jump = 1. The jumps of the second and the third set are 10 and 100 respectively. The exact size of these two jumps is not critical; other numbers such as 20 and 50 or 250 and 500, etc., could have been chosen; what we wish to illustrate is three levels of information: all values, every tenth value and every hundredth value, where these three procedures

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

correspond to detailed sampling, frequent sampling, and infrequent sampling. Obviously, to keep the number of observations the same, the interval of the second set is longer than the first and the third is longer than the second.

We next ask the fundamental question: what methods are available to the decision scientist to allow him/her to distinguish whether a data set of observations is generated by a deterministic or random function? Scientists from various backgrounds have researched this question extensively. Key references are Grassberger and Procaccia (1983), Takens (1985) and Brock, Hsieh and LeBaron (1992). For our purposes, we will briefly exposit the two main techniques, namely, the correlation dimension and the BDS tests. Then, in Tables 1 and 2, we will present the results of these two tests.

The correlation dimension was originally proposed by Grassberger and Procaccia (1983). Suppose that we are given a time series of price changes $\{x(t) : t = 0, 1, 2, \ldots, T\}$. Suppose that T is large enough so that a strange attractor has begun to take shape. Use this time series to create pairs, i.e. $x(t) \sim \{[x(t), x(t+1)] : t = 0, 1, 2, ..., T\}$ and then triplets and finally M-histories, $(t) \sim \{[x(t), ..., x(t+M-1)] : t = 0, 1, 2, \ldots, T\}$. In other words, we convert the original time series of singletons into vectors of dimension 2, 3, ..., M. In generating these vectors, we allow for overlapping entries. For example, if $M \sim 3$, we have a set of the form $\{[x(0), x(1), x(2)], [x(1), x(2), x(3)], \ldots, [x(T-2), x(T-1), x(T)]\}$. Such a set will have $(T+1)-(M-1)$ vectors. Mathematically, the process of creating vectors of various dimension from the original series is called an embedding.

Suppose that for a given embedding dimension, say M, we wish to measure if these M-vectors fill the entire M-space or only a fraction. For a given $\varepsilon > 0$, define the correlation integral, denoted by $C^M(\varepsilon)$, to be:

$$\begin{aligned} C^M(\varepsilon) &= \frac{\text{the number of pairs } (s,t) \text{ whose distance } \|x^M(s) - x^M(t)\| < \varepsilon}{T^2 M} \\ &= \frac{\text{the number of } (s,t), 1 \le t, s \le T, \|x^M(s) - x^M(t)\| < \varepsilon}{T^2 M} \end{aligned} \tag{5.5}$$

where $T_M = (T+1)-(M-1)$, and as before $x^M(t) = [x(t), x(t+1), ..., x(t+M-1)]$.

Observe that $\|\cdot\|$ in 5.5 denotes vector norm. Using the correlation integral, we can define the correlation dimension for an embedding dimension M as:

$$D^M = \lim_{\substack{\varepsilon\to 0 \\ T\to\infty}} \frac{\ln C^M(\varepsilon)}{\ln(\varepsilon)} \tag{5.6}$$

In 5.6 $\ln$ denotes natural logarithm. Finally, the correlation dimension D is given by:

$$D = \lim_{M\to\infty} D^M \tag{5.7}$$

We remark that technical accuracy requires that D^M in is a double limit, first in terms of $T \to \infty$ and then in terms of $\varepsilon \to 0$. However, in practice T is usually given and it is impossible to increase it to infinity. Thus the limit $T \to \infty$ is meaningless in practice and moreover M is practically bounded by T. Therefore, we only consider the limit $\varepsilon \to 0$ in (5.6).

Table 5.1 collects the results for correlation dimension analysis. Observe that there are three key columns of results corresponding to ε, 0.5ε, and 0.1ε. These three columns attempt to numerically illustrate the limiting process in (5.6). Observe also that we offer seven rows for various embedding dimensions 2, 3, 4, 5, 10, 15, and 20.

The results clearly demonstrate that for jump = 1, the correlation dimensions analysis detects the deterministic structure since the numbers for sample 1 and for each ε, 0.5ε, and 0.1ε are small and converge to a number between 1 and 2; (see column 0.1å for sample 1). On the other hand, the numbers for sample 10 are larger as ε decreases to 0.5ε and 0.1ε, these numbers do not converge; (see column 0.1ε for sample 10 as the number starts from 1.3611 and grows beyond 3.5810 to become indeterminate. Finally, for sampling every hundredth, the numbers of the correlation dimension are even larger and diverge sooner than for those of jump = 10. To summarize, a decision scientist would conclude that observations sampled from the x-variable of the Lorenz equations constitute a random set.

The second test we perform is the BDS, extensively presented in Brock, Hsieh and LeBaron (1991) and Brock, Dechert, Scheinkman and LeBaron (1996) These authors report that for an independent and identically distributed random process and for fixed M-histories and $\varepsilon > 0$,

$$C^M(\varepsilon, T) \quad \to \quad [C^1(\varepsilon)]^M \text{ as } T \to \infty \tag{5.8}$$

They further report that as T approaches infinity,

$$\sqrt{T}\{C^M(\varepsilon, T) - [C^1(\varepsilon, T)]^M\} \to N(0, \sigma^2(\varepsilon, T)), \tag{5.9}$$

where N denotes a normal distribution with mean zero and variance $\sigma^2(\varepsilon, T)$. From the above two equations 5.8 and 5.9, it is concluded that

$$\frac{\sqrt{T}\{C^M(\varepsilon, T) - [C^1(\varepsilon, T)]^M\}}{\sigma^M(\varepsilon, T)} \to N(0, 1), \tag{5.10}$$

Table 5.2 has only two values which lie in the [-1.96, 1.96] interval of the standardized normal distribution. These are the values: -0.0107 corresponding to jump 10, 1.0ε and $M = 3$ and -1.6259 corresponding to jump 100, 1.0ε and $M = 15$. For all the other values, we reject the null hypothesis of randomness. Note that the BDS does not claim that our three samples of 5000 observations are deterministic because its alternative to the null hypothesis is not well specified. Thus, a researcher could not conclude that the underlying structure of our samples is deterministic; he or she could only reject, with two exceptions, the null hypothesis.

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

Times of å	1,0ε			0,5ε			0,1ε		
Value of ε	0.2205	0.2183	0.2160	0.11025	0.10915	0.108	0.02205	0.02183	0.0216
Sample	1	10	100	1	10	100	1	10	100
Embedding Dimension									
2	0.66157	0.90593	0.90765	0.84393	1.13420	1.17630	1.14340	1.36110	1.49650
3	0.88411	1.35200	1.36670	1.03870	1.59770	1.77020	1.32140	1.77130	2.24430
4	1.05840	1.76940	1.82400	1.18270	2.01190	2.36340	1.44140	2.08380	3.00400
5	1.17740	2.13320	2.28270	1.27740	2.36910	2.95770	1.52080	2.36430	3.84110
10	1.60950	3.99840	4.59000	1.69210	4.18990	5.93220	1.83130	3.58190	N/A
15	2.00860	5.87520	6.85070	1.99530	5.97950	N/A	2.00990	N/A	N/A
20	2.35560	7.68720	9.38890	2.26830	7.06120	N/A	2.16770	N/A	N/A

Table 5.1: Dimension analysis D^M for x_t (5000 observations generated from Lorenz Equation for step 0.1). Note: ε=0,2205 for sample=1; ε=0,2183 for sample=10; ε=0,2160 for sample=100.

Times of ε	1,0ε			0,5ε			0,1ε		
Value of ε	0.2205	0.2183	0.2160	0.11025	0.10915	0.108	0.02205	0.02183	0.0216
Sample	1	10	100	1	10	100	1	10	100
Embedding Dimension									
2	1,5672E+2*	-2.4762*	-5.7504*	2,7225E+2*	2,1428E+1*	-5.9515*	6,1123e+2*	1,4718e+2*	-3.3343*
3	1,6288E+2*	**-1.0658E-02**	-5.0498*	4,3919E+2*	4,7617E+1*	-5.3501*	3,0973e+3*	4,8539e+2*	-1.9821*
4	1,9461E+2*	4,8161*	-4.1820*	7,9437E+2*	7,1756E+1*	-4.6143*	2,0468e+4*	1,7420e+3*	-4.0175*
5	2,5816E+2*	1,2546E+1*	-3.7614*	1,6643E+3*	1,0937E+2*	-4.1692*	1,7594e+5*	7,1521e+3*	-1.34E+01*
10	1,5363E+3*	2,2178E+1*	-2.8503*	1,2980E+5*	5,1375E+2*	-2.7778*	2,4482e+10*	3,2205e+7*	-1.12E+01*
15	1,2904E+4*	2,6654E+1*	**-1.6259**	2,0387E+7*	3,1108E+3*	-6.6069*	9,0690e+15*	-4.8504*	-5.2183*
20	1,3955E+5*	2,6288E+1*	2.6727*	4,0968E+9*	1,0592E+3*	-3.7957*	4,3345e+21*	-2.6627*	-2.8917*

Table 5.2: BDS Test for for x_t (5000 observations generated from Lorenz Equation for step 0.1). Note: ε=0,2205 for sample=1; ε=0,2183 for sample=10; ε=0,2160 for sample=100.
* Reject null hypothesis of randomness

5.5 Evaluation and conclusion

This paper has reviewed the methodological foundations of deterministic and random modeling and argued that determinism remains the scientific goal of any investigation. Recent papers in economic theory as in Brock and Hommes (1998), Lux (1998), Malliaris and Stein (1999), Chiarella, Dieci and Gardini (2000), among others, show that even if the time series behavior of a given model looks like random, its underlying structure may still be deterministic.

Suppose that the underlying relationships are exact. What could account for our inability to detect such a structure and then to build models that would make such a structure explicit. In this paper, we make a contribution by demonstrating that the currently available techniques for distinguishing between deterministic and random systems are not adequate. The correlation dimension performs well when every value of the Lorenz equation is sampled, but does poorly when the jump increases to 10 and then to 100. This illustrates that unless, in the real world, we can record information at high frequencies rather than at prespecified intervals, say end

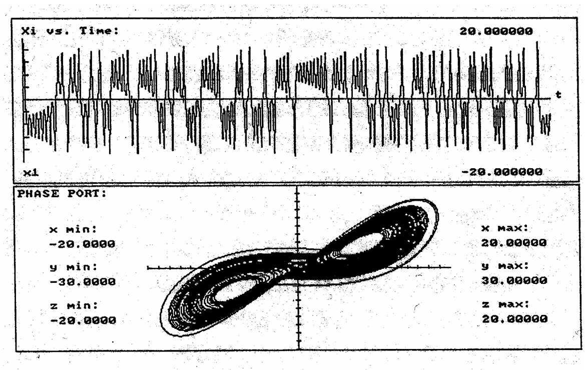

Figure 5.1: Rules of 4-day period

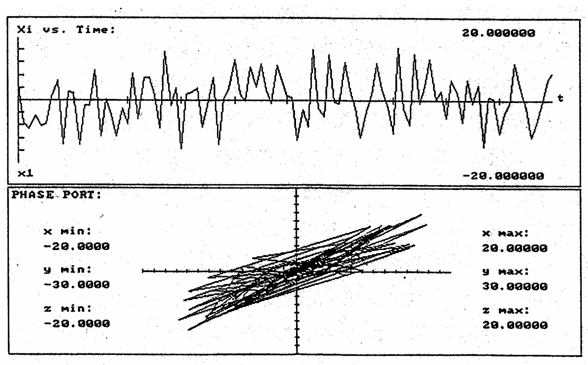

Figure 5.2: Rules of 4-day period

of the day, weekly, monthly, etc., we are bound to lose the underlying structure. Our experiment shows that infrequent sampling misses the deterministic relationship. Of course, data limitations may not allow a scientist to perform the tests we used. For example using annual or quarterly data, one does not have enough observations to do dimension and BDS analysis. There is evidence provided by Ramsey, Sayers and Rothman (1990) that dimension calculations using small data sets are biased. However, the availability of massive data is rapidly becoming a reality and such data can be conveniently analyzed by the techniques demonstrated. The BDS does very well rejecting randomness in our sample, but cannot specify the alternative.

Our overall conclusion is simply this: since WWII, the scientific pendulum in general and in management science, financial economics and forecasting in particular, has been pulled away from determinism and brought towards stochasticity. But

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

such stochasticity has not fully enriched our understanding of the real world simply because what drives randomness often cannot be anticipated. Chaotic dynamics is not a totally new methodology, but rather a new way of affirming order, rationality and exactness despite the seeming disorderly, unpredictable and random behavior of certain variables. This discovery of chaotic dynamics and our increasing understanding of the Lorenz equation offer valid alternatives to randomness.

Bibliography

[1] Brock, W. and A.G. Malliaris (1989), Differential Equations, Stability and Chaos in Dynamic Economics, Amsterdam: Elsevier Publishing.

[2] Brock, W., D. Hsieh and B. LeBaron, (1992), Nonlinear Dynamics, Chaos and Instability: Statistical Theory and Economic Evidence, Second Edition, Cambridge, Massachusetts: The MIT Press.

[3] Brock, W.A , Dechert , W.D. and Scheinkman, J.A.and B. LeBaron (1996). "A Test for Independence Based on the Correlation Dimension," *Econometric Reviews*, 15, pp.197- 235.

[4] Brock, W.A. and Hommes , C.H. (1998). "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model," *Journal of Economic Dynamics and Contro*l, 22, 1235- 1274.

[5] Campbell, John, A. Lo and A. MacKinlay (1997), The Econometrics of Financial Markets, Princeton: Princeton University Press.

[6] Chiarella, C., Dieci, R. and Gardini, L. (2000), "Speculative Behavior and Complex Asset Price Dynamics," *Journal of Economic Behavior and Organization*, 49, pp.173-97.

[7] Corazza, M. and Malliaris A.G.(2002). "Multi-Factuality in Foreign Currency Markets," *Multinational Finance Journal*, 6, 65-98.

[8] Devaney, R., (1986), An Introduction To Chaotic Dynamical Systems, Menlo Park, California: Benjamin/Cummings Publishing.

[9] Eckmann, J. and D. Ruelle, (1985), "Ergodic Theory of Chaos and Strange Attractors," *Review of Modern Physics*, Vol. 57, pp. 617-656.

[10] Grassberger, P. and I. Procaccia, (1983), "Measuring the Strangeness of Strange Attractors," *Physics*, Vol. 9-D, pp. 189-208.

[11] Guckenheimer, J. and P. Holmes, (1983), Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, New York: Springer-Verlag.

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

[12] Hsieh, D.A. (1989). "Testing Nonlinear Dependence in Daily Foreign Exchange Rates" *Journal of Business*, pp. 339-368.

[13] Hsieh, D.A. (1991). "Chaos and Nonlinear Dynamics: Application to Financial Markets" The *Journal of Finance*, 46, pp. 1839-1877.

[14] Koçak, H., (1989), Differential and Difference Equations through Computer Experiments, New York: Springer-Verlag.

[15] Kyrtsou, C. and Terazza , M. (2002), "Stochastic Chaos or ARCH Effects in Stock Series? A Comparative Study," *International Review of Financial Analysis*, 11, pp. 407-31.

[16] Kyrtsou, C. and Terazza , M. (2003), "Is It Possible to Study Chaotic Dynamics and ARCH Behavior Jointly? Application of a Noisy Mackey-Glass Equation with Heteroskedastic Errors to the Paris Stock Exchange Returns Series," *Computational Economics*, 21, pp. 257-75.

[17] Lo, A. (1991), "Long-term Memory in Stock Market Prices," *Econometrica*, 59, pp. 1279-1313.

[18] Lo, A. and A. Craig MacKinlay (1999), A Non-Random Walk Down Wall Street, Princeton: Princeton University Press.

[19] Lorenz, E., (1963), "Deterministic Non-Periodic Flows," *Journal of Atmospheric Sciences*, 20, pp. 130-141.

[20] Lux, T. (1998), "The Socio-dynamics of Speculative Markets: Interactive Agents, Chaos and the Fat Tails of Returns Distributions," *Journal of Economic Behavior and Organization*, 33, pp. 143-165.

[21] Malliaris A.G. and Stein, J.L (1999), "Methodological Issues in Asset Pricing: Random Walk or Chaotic Dynamics," *Journal of Banking and Finance*, 23, pp. 1605-1635.

[22] Malliaris, M., (1993), "Modeling the Behavior of the S&P 500 Index: A Neural Network Approach," *Neurovest*, Vol. 3, No. 3, May/June, 1995, pp. 16-21.

[23] Malkiel, B (2003), A Random Walk Down Wall Street, 8th Edition, New York: Norton.

[24] May, R., (1976), "Simple Mathematical Models With Very Complicated Dynamics," *Nature*, Vol. 261, pp. 459-467.

[25] Ramsey, J.B., C. Sayers and P. Rothman, (1990), "The Statistical Properties of Dimension Calculations Using Small Data Sets: Some Economic Applications," *International Economic Review*, 31, pp. 991.

in C. Kyrtsou and C. Vorlow (eds): Progress in Financial Markets Research, NOVA Publishers, New York, 2010, in Press.

[26] Rasmussen, D.R, and E. Mosekilde, (1988), "Bifurcations and Chaos in a Generic Management Model," *European Journal of Operational Research*, 35, pp. 80-88.

[27] Ruhla, C., (1992), The Physics of Chance: From Plaise Pascal to Niels Bohr, New York: Oxford University Press.

[28] Scheinkman, J.A. and LeBaron , B. (1989), "Nonlinear Dynamics and Stock Returns," *Journal of Business*, 62, pp. 311-337.

[29] Sengupta, J.K. and Zheng, Y. (1995), "Empirical Tests of Chaotic Dynamics in Market Volatility," *Applied Financial Economics*, 5, pp. 291-300.

[30] Sparrow, C., (1982), The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors, New York: Springer-Verlag.

[31] Sterman, S., (1989), "Modeling Managerial Behavior: Misperceptions of Feedback in a Dynamic Decision Making Experiment," *Management Science*, 35, pp. 321-339.

[32] Takens, F., (1985), "Distinguishing Deterministic and Random Systems," in G. Borenblatt, G. Iooss and D. Joseph, editors, Nonlinear Dynamics and Turbulence, Boston: Pitman, pp. 315-333.