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Interest rates and inflation *

A continuous time stochastic approach

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This paper investigates the theoretical foundations of Fisher's equation which expresses the nominal interest rate as the sum of the real interest rate and the expected rate of inflation. To emphasize Fisher's (1930) original formulation and Sargent's (1973) recent suggestion that nominal interest rates and inflation are simultaneously determined rather than having the causation go from inflation to interest rates, we develop a two-equation continuous time stochastic model to build a more solid theoretical foundation of Fisher's equation. Assuming that the nominal interest rate and the rate of inflation follow Itô processes we derive an Itô equation that allows us to express and compute the expected real interest rate and its volatility. These two equations generalize the traditional Fisher equation and an illustration using US long data from 1865–1972 shows the usefulness of our results.

1. Introduction

Keynes (1930, pp. 198–210) observed that A.H. Gibson, a businessman, had written several articles during the 1920's about the fact that interest rates exhibited a high correlation with inflation. Keynes called this empirical relationship between interest rates and inflation the 'Gibson paradox' because it appeared contradictory to the classical monetary theory prediction that the interest rate is independent of the price level. Fisher (1930) proposed his, by now famous, hypothesis to explain the Gibson paradox. Fisher suggested that the nominal interest rate, R, is the sum of the real rate of return on assets, r, with such return fixed in real terms and the expected rate of inflation denoted as $\mathrm{E}(\mathrm{d}P/\mathrm{d}t)(1/P)$, that is,

$$R = r + \mathrm{E}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\frac{1}{P}\right), \tag{1}$$

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where P denotes some index of the economy's price level, such as the consumer price index, or the GNP implicit price deflator. Tanzi (1980) reviews the earlier literature and Tobin (1987) overviews Fisher's ideas on this topic, and clarifies three, often overlooked, important issues about (1).

First, in its simplest form, Fisher's equation connects the nominal interest rate to the real interest rate and the actual inflation i.e. $R = r + (\mathrm{d}P/\mathrm{d}t)\ (1/P)$ and as such it is first of all an identity relating the unobservable value of real interest rate r to two observable variables, the nominal interest rate and the rate of actual price inflation.

Second, Fisher viewed (1) as an equilibrium condition in the financial markets and for this purpose he replaced the actual inflation $(\mathrm{d}P/\mathrm{d}t)(1/P)$ by the expected inflation $\mathrm{E}((\mathrm{d}P/\mathrm{d}t)\ (1/P))$. In such a formulation, (1) relates one observable variable, i.e. the nominal interest rate R, to two unobservable ones, which makes (1) impossible to test empirically.

Third, Fisher recognized that in the longer run, steady-state equilibrium would also be characterized by equality of actual and expected inflation, $(\mathrm{d}P/\mathrm{d}t)\ (1/P) = \mathrm{E}((\mathrm{d}P/\mathrm{d}t)\ (1/P))$. Assuming such an equilibrium condition allows one to use data for the two observables, nominal interest rates and inflation, to compute the real rate of interest.

Sargent (1973) in his comprehensive article on this topic concludes that Fisher's explanation of the paradox is inadequate it posits an unidirectional influence flowing from inflation to interest rates. He suggests that an adequate theory claiming to explain the Gibson paradox must account for the fact that in a rational expectations environment interest rates and inflation are being mutually determined.

Tobin (1987) claims that it was not actually Fisher but some of his followers who frequently cited (1) in support of a complete and prompt transmission of inflation into nominal interest rates. Such view usually assumes the constancy of the real interest rate and empirically tests for market efficiency articulated by Fama (1975, p. 269): 'If the inflation rate is to some extent predictable, and if the one-period equilibrium expected real return does not change in such a way as to exactly offset changes in the expected rate of inflation, then in an efficient market there will be a relationship between the one-period nominal interest rate observed at a point in time and the one-period rate of inflation subsequently observed'.

Tobin (1987, p. 375) writes that 'Fisher's view throughout his career was quite different. For one thing, neither Fisher's theory of interest nor his reading of historical experience suggested to him that equilibrium real rates of interest should be constant. Fisher's original views as well as Sargent's (1973) suggestions. In section 3 we use long term data from Friedman and Schwartz (1982) to illustrate the theoretical results. A brief summary and the main conclusions are summarized in the last section.

2. A continuous time stochastic approach

To follow up on Tobin's claims about Fisher's actual view of (1) and also Sargent's obvious suggestion of the mutual interdependence inflation and interest rates and simultaneously to do it in a way that the market methodology is preserved we propose the following model. Assume that nominal interest rates and inflation follow Itô's stochastic differential equations given by

$$\frac{\mathrm{d}Q}{Q} = \mu_R(t, R)\ \mathrm{d}t + \sigma_R(t, R)\ \mathrm{d}Z_R, \tag{2}$$

$$\frac{\mathrm{d}P}{P} = \mu_P(t, P)\ \mathrm{d}t + \sigma_P(t, P)\ \mathrm{d}Z_P, \tag{3}$$

with

$$dZ_R\, dZ_P = \rho_{RP}\, dt. \tag{4}$$

Note that (2) describes the nominal return of an asset per unit of time as an Itô process. For example, (2) may describe the nominal return of a bond valued at $\$Q$ at time t. Therefore, μ_R denotes the instantaneous nominal interest rate, that is, $\mu_R \equiv R$, and σ_R denotes its instantaneous volatility. In (3), inflation is expressed again as an Itô process with μ_P denoting the instantaneous rate of expected inflation, that is, $E(dP/dt)(1/P) \equiv \mu_P$, and σ_P denoting inflation volatility. Observe that both nominal interest rates in (2) and inflation in (3) are shocked by random forces denoted by dZ_R and dZ_P respectively. These variable denote standardized Wiener processes and in order to implement Fisher's views and Sargent's suggestion of the mutual interdependence of inflation and interest rates we postulate in (4) that these Wiener processes are correlated, with $\rho_{RP} \neq 0$. The methodological and mathematical foundations of continuous time stochastic modelling are presented in Malliaris and Brock (1982) and Merton (1975, 1982).

Next we define the real value of an asset, denoted by q, as in Fischer (1975)

$$q = Q/P, \tag{5}$$

and ask the question: what is the behavior of q in view of the two processes in (2) and (3)? Itô's lemma and calculations yield the answer to this question, namely,

$$dq/q = d\left(\frac{Q}{P}\right)\Big/\left(\frac{Q}{P}\right) = \left[\mu_R - \mu_P - \sigma_R\sigma_P\rho_{RP} + \sigma_P^2\right] dt + \sigma_R\, dZ_R - \sigma_P\, dZ_P. \tag{6}$$

Equation (6) describes the proportional change in the real rate of interest as an Itô process and is consistent with Fisher's view of the nonconstancy of the real rate of interest. Taking the expectation and variance of (6) we get

$$E\left(\frac{dq}{q}\right) = \mu_R - \mu_P - \sigma_R\sigma_P\rho_{RP} + \sigma_P^2, \tag{7}$$

$$\operatorname{Var}\left(\frac{dq}{q}\right) = \sigma_R^2 - 2\sigma_R\sigma_P\rho_{RP} + \sigma_P^2. \tag{8}$$

The pair of eqs. in (7) and (8) generalize in a mathematically precise and an economically important way Fisher's equation in (1). Mathematically, $E(dq/q)$ measures the real return and therefore (7) generalizes Fisher's equation (1). Since $E(dq/q) \equiv r$ then (7) can be written as

$$r = \mu_R - \mu_P - \sigma_R\sigma_P\rho_{RP} + \sigma_P^2, \tag{9}$$

which reduces to Fisher's equation if we assume that $\sigma_R = \sigma_P = 0$, i.e., if we assume that both the volatilities of nominal interest rates and inflation are zero. In other words, our modelling has enabled us to generalize Fisher's equation. More specifically, since we have already observed that $\mu_R = R$ and $E(dP/dt)(1/P) \equiv \mu_P$, making these two substitutions and rearranging terms, (9) can be written as

$$R = r + E\left(\frac{dP}{dt}\frac{1}{P}\right) + \sigma_R\sigma_P\rho_{RP} - \sigma_P^2. \tag{10}$$

This last equation illustrates that Fisher's eq. (1) is a special case of (10) obtained when volatilities are ignored. It is worth recalling at this point that Fisher (1930, 1963) and numerous researchers of this topic realize that a more accurate relation for (1) can be obtained from

$$1+R=(1+r)\left[1+\mathrm{E}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\frac{1}{P}\right)\right], \tag{11}$$

where $(1+R)$ denotes the nominal increase of a \$1 debt, $(1+r)$ measures the real productivity increase and $1+\mathrm{E}(\mathrm{d}P/\mathrm{d}t)(1/P)$ denotes the increase in value due to expected inflation. From (11) we conclude that

$$R=r+\mathrm{E}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\frac{1}{P}\right)+r\mathrm{E}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\frac{1}{P}\right), \tag{12}$$

which is a more accurate discrete time version of (1). The reason that (1) is stated as Fisher's equation instead of (12) is because the term $r\mathrm{E}(\mathrm{d}P/\mathrm{d}t)(1/P)$ is arithmetically insignificantly small. Our term $\sigma_R\sigma_P\rho_{RP}$ in (10) actually captures the impact of $r\mathrm{E}(\mathrm{d}P/\mathrm{d}t)(1/P)$ (12). But even if the more accurate version of Fisher's equation in (12) is used, still the continuous time modelling yields a generalization because of the extra term σ_P^2 in (9) and, more importantly, because it yields one whole additional equation (8) describing the variance of the real interest rate. In other words, we need not assume as fisher's followers claim that the real interest rate is constant and by allowing it to vary, the square root of equation (8) describes the time path of its volatility. There is no a priori reason why the variance of the real rate of return should be zero in (8). If (8) is not zero, then the real rate of interest is not constant and equation (8) describes the volatility of the real rate of interest as the sum of the volatility of the nominal interest rate and inflation, less two times their covariance.

3. U.S. data 1865–1972

Friedman and Schwartz (1982) have collected and analyzed extensively annual US data for the period 1865–1972. We use their data for our two variables: short-term nominal interest rates on commercial paper and inflation. Simple calculations show that the average nominal interest rate and its volatility (standard deviation) are $\mu_R=0.042183$ and $\sigma_R=0.020949$, while the average rate of inflation and its volatility are $\mu_P=0.013084$ and $\sigma_P=0.05269$. Assuming that expected inflation was correctly anticipated and therefore that expected inflation was equal to actual inflation, Fisher's equation (1) yields an average real interest rate for the US during 1865–1972 given by

$$r=R-\mathrm{E}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\frac{1}{P}\right)=R-\frac{\mathrm{d}P}{\mathrm{d}t}\frac{1}{P}=0.042183-0.013084=0.029099. \tag{13}$$

Next, we wish to show the numerical results obtained from eqs. (7) and (8) derived using stochastic calculus techniques. Observe that we need to compute ρ_{RP} from (4). To accomplish this, first use the values of $\mu_R=0.042183$, $\sigma_R=0.020949$, $\mu_P=0.013084$, and $\sigma_P=0.05269$ to write (2) and (3) as

$$\frac{\mathrm{d}Q}{Q}=0.042183\,\mathrm{d}t+0.020949\,\mathrm{d}Z_R, \tag{14}$$

$$\frac{\mathrm{d}P}{P}=0.013084\,\mathrm{d}t+0.05269\,\mathrm{d}Z_P. \tag{15}$$

Next, use the actual time series of nominal interest rates and (14) to solve for $\mathrm{d}Z_R$. Similarly, obtain a time series for errors $\mathrm{d}Z_P$ from (15), using the actual annual inflation or deflation. For purposes of diagnostics we checked and confirmed that $\mathrm{E}(\mathrm{d}Z_R) = \mathrm{E}(\mathrm{d}Z_P) = 0$ and $\mathrm{Var}(\mathrm{d}Z_R) = \mathrm{Var}(\mathrm{d}Z_P) = 1$. From these two time series $\mathrm{d}Z_R$ and $\mathrm{d}Z_P$, obtained as just described, we can readily compute their correlation coefficient ρ_{RP}. It is found to be $\rho_{RP} = -0.19681$. The negative sign indicates that the various shocks that have affected prices positively had an average negative impact on nominal interest rates. This finding cannot be supported by the efficient market hypothesis but is nevertheless an empirical phenomenon of this period during which inflation was highly volatile while nominal interest rate were much more stable and responded to such inflation volatility with lags.

With these calculations completed, (9) yields

$$\begin{aligned} r &= \mu_R - \mu_P - \sigma_R \sigma_P \rho_{RP} + \sigma_P^2, \\ &= (0.042183) - (0.013084) - (0.020949)(0.05269)(-0.19681) + (0.05269)^2, \\ &= 0.032092476, \end{aligned} \tag{16}$$

while the variance of the real interest rate is given by

$$\begin{aligned} \mathrm{Var}\ r &= \sigma_R^2 - 2\sigma_R \sigma_P \rho_{RP} + \sigma_P^2, \\ &= (0.20949)^2 - 2(0.020949)(0.05269)(-0.19681) + (0.5269)^2, \\ &= 0.003649576. \end{aligned} \tag{17}$$

Note that (16) slightly overestimates the real interest rate obtained from the simple, traditional Fisher equation in (13). The reason is that inflation's volatility during this period was very high and therefore the term σ_P^2 is not negligible. Also, although the term $\sigma_R \sigma_P \sigma_{RP}$ is very small, the fact that ρ_{RP} is negative also ends up contributing to the real interest rate. On the basis of one application of the generalized Fisher equation one could not claim that (13) underestimates the real interest rate. However, it is illustrative that (16) gives a much more accurate estimation of the real interest rate since it contains four terms instead of just two for the simple Fisher equation. In other words, the generalization of Fisher's equation having four terms in (16) instead of only two terms in (13) is not trivial. It would have been arithmetically trivial, although still interesting analytically, if the two extra terms in (16) were numerically always negligible.

Finally, observe that taking the square root of (17) we obtain that the volatility of the real interest rate during 1865–1972 was 0.060411721. This implies that the real interest rate was not a constant during this period. Thus equation (17) supplies us with useful information which is not available from Fisher's elementary theory.

4. Conclusions

This paper investigates the theoretical foundations of Fisher's equation which expresses the nominal interest rate as the sum of the real interest rate and the expected rate of inflation. Following Fisher, we assume that an equilibrium condition in financial markets implies that expected inflation equals actual inflation and therefore one can study the impact of inflation (expected and actual) on nominal interest rates).

To emphasize Fisher's (1930) original formulation and Sargent's (1973) recent suggestion that nominal interest rates and inflation and simultaneously determined rather than having the causation go from inflation to interest rates, we develop a two-equation continuous time stochastic model to build a more solid theoretical foundation of Fisher's equation. Assuming that the nominal interest rate and the rate of inflation follow Itô processes we derive an Itô equation that allows us to express and compute the expected real interest rate and its volatility. These two equations generalize the traditional Fisher equation and an illustration using US long data from 1865–1972 shows the usefulness of our results.

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