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Certain Selecting and Underreporting Processes

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Abstract—In this paper, we present a model for promotional advertising and underreporting of incomes. The model is based on the integral part of the product of a discrete random variable with a continuous uniformly distributed random variable.

Keywords—Integral part, Promotional advertising, Income underreporting.

1. INTRODUCTION

Let X and Y be continuous nonnegative random variables. The model

$$Y = UX, \tag{1.1}$$

where U is distributed independently of X in $(0,1)$, is known to have important applications in different fields. In income distribution analysis, X represents true income and Y reported income [1]. In inventory decision making, X represents demand for an item within a unit time interval and Y item units in stock within the same time interval [2]. Moreover, in discounting cash flows, X represents a payment to be paid at some future time and Y the present value of the payment [3,4].

Let X be a discrete random variable with values in $\{1, 2, \dots\}$ and U distributed independently of X in $(0,1)$. A stochastic multiplicative model as in (1.1) appropriately modified to account for discrete random variables X, Y is given by

$$Y = [UX], \tag{1.2}$$

where $[UX]$ denotes the integral part of UX. Krishnaji [5], though not referring to financial models, has used (1.2), with U uniformly distributed in $(0,1)$, to establish a characterization of a zero-truncated Yule distribution. If $P_X(z)$ is the probability generating function of the random variable X, and U is uniformly distributed in $(0,1)$, then

$$P_Y(z) = \frac{1}{1-z} \int_z^1 \frac{P_X(\psi)}{\psi} d\psi, \qquad |z| \le 1, \tag{1.3}$$

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is the probability generating function of the random variable Y [5]. The present paper proposes a model which is an extension of model (1.2). Properties and applications of the model are established.

From a theoretical point of view, the proposed model can be considered, in some sense, as a discrete analogue of a particular case of a continuous model proposed by Artikis and Malliaris [6]. The model studied in this paper is based on the concept of "the integral part" and hence the model is not the direct discrete analogue of Artikis and Malliaris' continuous model. Moreover, the proposed discrete model is applied to underreporting of income and promotional advertising, whereas Artikis and Malliaris' continuous model is applied to financing new investments.

2. THEORETICAL RESULTS

Consider the random sum

$$X_1 + X_2 + \cdots + X_N,$$

where $\{X_n : n = 1, 2, \dots\}$ is a sequence of independent discrete random variables with values in $\{1, 2, \dots\}$, each with probability generating function $P_X(z)$, and N is a discrete random variable taking values in $\{1, 2, \dots\}$ with probability generating function $P_N(z)$, independent of $\{X_n : n = 1, 2, \dots\}$. Let V, W be independent discrete random variables taking values in $\{1, 2, \dots\}$ with probability generating functions $P_V(z), P_W(z)$, respectively, independent of $\{X_n : n = 1, 2, \dots\}$ and N. Furthermore, let U be a continuous random variable distributed independently of $\{X_n : n = 1, 2, \dots\}, N, V$ and W in $(0, 1)$.

Consider the model

$$S = V + [U(X_1 + X_2 + \cdots + X_N + W + 1)]. \tag{2.1}$$

The distribution of the above model is, in general, very complicated. This section explicitly derives the distribution of a particular case of model (2.1).

THEOREM 1. *Let U be uniformly distributed in* $(0, 1)$. *Then*

$$P_S(z) = P_V(z) \exp\left\{-\int_z^1 \frac{1 - P_V(\psi) P_N(P_X(\psi))}{1 - \psi}\, d\psi\right\} \tag{2.2}$$

is the probability generating function of S if and only if

$$S \stackrel{d}{=} V + [U(X_1 + X_2 + \cdots + X_N + S + 1)] \tag{2.3}$$

where $\stackrel{d}{=}$ means equality in distribution.

PROOF. Only sufficient conditions will be proved since the necessity can be proved by reversing the argument. Using probability generating functions in (2.3), we get the integral equation

$$P_S(z) = P_V(z) \frac{1}{1 - z} \int_z^1 P_N(P_X(\psi)) P_S(\psi)\, d\psi. \tag{2.4}$$

Since $P_S(z)$ and $P_V(z)$ are probability generating functions, they are differentiable for $0 < z < 1$. Multiplying both sides of (2.4) by

$$\frac{1 - z}{P_V(z)},$$

such that $P_V(z) \neq 0$ and differentiating, we get the differential equation

$$P_S'(z) = \left\{\frac{1 - P_V(z) P_N(P_X(z))}{1 - z} + \frac{P_V'(z)}{P_V(z)}\right\} P_S(z) \tag{2.5}$$

with the condition $P_S(1) = 1$. Integrating in (2.5) we get

$$P_S(z) = P_V(z)\exp\left\{-\int_z^1 \frac{1 - P_V(\psi)P_N(P_X(\psi))}{1-\psi}\,d\psi\right\}. \tag{2.6}$$

From (2.6), it follows that $P_S(z)$ is the product of two probability generating functions. The first term in the right hand side of (2.6) is the probability generating function of the random variable V and the second term is a self-decomposable probability generating function. The class of discrete self-decomposable distributions is closed under convolution. Moreover, the discrete self-decomposable distributions are unimodal [7]. Assuming that $P_V(z)$ is self-decomposable, it follows that $P_S(z)$ is self-decomposable, and hence, the corresponding distribution is unimodal. The existence of a unique mode is clearly essential, since the presence of multimodality introduces a degree of localized ambiguity into the decision process. The class of discrete self-decomposable distributions includes many distributions which are important in statistics, actuarial science and in financial economics.

The Geometric distribution, the Poisson distribution, the mixture of Poisson distribution whose mixing distribution is self-decomposable with positive support, and the stable distributions are examples of discrete self-decomposable distributions.

We consider two particular cases of the distribution of the random variable S. By putting

$$P_N(z) = z,$$

$$P_X(z) = z,$$

and

$$P_V(z) = z$$

in (2.6) we get that

$$P_S(z) = z\exp\left\{(z-1) + \frac{1}{2}(z^2-1)\right\}$$

which is the probability generating function of the shifted Hermite distribution with parameters 1 and 1/2, see [8]. Moreover, by putting

$$P_N(z) = \frac{\lambda z}{1-(1-\lambda)z}, \qquad 0 < \lambda < 1,$$

$$P_X(z) = \frac{\mu z}{1-(1-\mu)z}, \qquad 0 < \mu < 1,$$

and

$$P_V(z) = \frac{pz}{1-qz}, \qquad 0 < p < 1, \quad q = 1-p,$$

with $p = \lambda\mu$ and $\lambda\mu < 1/2$, in (2.6) we get that

$$P_S(z) = \frac{pz}{1-qz}\left(\frac{p}{1-qz}\right)^{(q-p)/q^2}\exp\left\{\frac{1}{q}\left[\frac{pz}{1-qz}-1\right]\right\},$$

which is the product of three probability generating functions. The first probability generating function belongs to the Geometric distribution with parameter p, the second belongs to the Polya Eggenberger distribution with parameters, q/p and $(q-p)/pq$, and the third belongs to the Polya Aeppli distribution with parameters $1/q$ and p, see [8].

3. APPLICATIONS OF THE MODEL

In this section, we illustrate how (2.1) and (2.3) arise in economics. Suppose that a manufacturing firm produces N products. The number of products N is random because every year new products are added and some old ones are eliminated from production. Denote by $X_n : n = 1, 2, \ldots,$ the total number of customers of products $1, 2, \ldots,$ who bought each at least one unit of a given product. Observe that X_n is measured in integral parts because the units denote number of customers. We assume that the random variables $X_n : n = 1, 2, \ldots,$ are independent because the firm's N products are dissimilar. Furthermore, suppose that the random variable $W + 1$ denotes the number of customers who bought in the past but not during the current year. From the total number of customers, both current and old, the firm uses the function U to select a certain subset. For example, suppose that the firm wishes to advertise to customers who are in a certain age bracket, say between 40 and 45 years old. In such a case, U will select from

$$X_1 + X_2 + \cdots + X_N + W + 1$$

the subset of the customers in such an age bracket.

Finally, let V denote potential customers who have not as yet purchased anything from the firm and who are targeted for promotional advertising. Under the scenario described, the random variable

$$S = V + [U(X_1 + X_2 + \cdots + X_N + W + 1)] \tag{3.1}$$

gives the total number of customers to receive promotional material. The usefulness of Theorem 1 is mainly demonstrated through equation (2.3) which is the sufficient condition for the evaluation of the distribution of S from the distributions of $V, U;\ X_n : n = 1, 2, \ldots,$ and N.

In income distribution analysis, the random variable S in (3.1) can be interpreted in the following way. Consider a firm whose total annual income is

$$V + X_1 + X_2 + \cdots + X_N + W + 1,$$

where V denotes income from securities, $X_1 + X_2 + \cdots + X_N$ income from N sales of a product, and $W+1$ income from other assets of the firm not used in its production process. We suppose that the random variables $V, X_n : n = 1, 2, \ldots,$ and W are measured in integral parts of some convenient unit. If the firm reports income V and reports a fraction U of income $X_1 + X_2 + \cdots + X_N + W + 1$, then the random variable S in (3.1) denotes the annual income reported by the firm.

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