ITÔ'S CALCULUS IN FINANCIAL DECISION MAKING*

A. G. MALLIARIS†

Abstract. This paper presents an introduction to Itô's stochastic calculus by stating some basic definitions, theorems and mathematical examples. Afterwards, the use of Itô's calculus in modern financial theory is illustrated by exposing a few representative applications. The main observation of this paper is that Itô's calculus which was developed from purely mathematical questions originating in Wiener's work has found unexpectedly important applicability in the theory of finance from the perspective of continuous time.

1. Introduction. Uncertainty and its modeling have played an important role in economic analysis. It is the purpose of this paper to demonstrate how certain relatively recent mathematical discoveries have enabled economists to formulate clearly and solve successfully several significant problems in financial economics. The mathematical theory we have in mind is known as Itô's calculus and includes stochastic integration, stochastic differentials, rules of stochastic differentiation, and stochastic differential equations.

The development of Itô's calculus was motivated by purely mathematical questions originating in Wiener's [35] work of 1923 on stochastic integrals. At no time did Itô or the other pioneering mathematicians working in this area realize the impact of their results on economic analysis. During the last two decades economists have used Itô's differential equation

\[ dX(t) = f(t, X(t)) \, dt + \sigma(t, X(t)) \, dZ(t) \]

(1)

to describe the behavior of stock prices, the stochastic rate of inflation, the stochastic spot rate of interest, and other economic variables. The availability of Itô's theory was not in itself sufficient for the formulation and solution of various economic problems. It had to be supplemented by appropriate breakthroughs in financial modeling. More on this issue will be said later on as the applications are explored. However, there is no doubt that Itô's calculus was necessary for the solution of several major economic problems.

Before we proceed to expound on Itô's calculus and to present some of its applications in financial economics, we should first say roughly why we consider stochastic models in economic analysis. In general, probability theory is introduced in economic analysis in cases when we are faced with an uncertain situation. As a researcher attempts to model the uncertain future behavior of an economic variable, probabilistic reasoning becomes appropriate by embedding a particular situation in a collection of like situations. Arrow [2] argues that the uncertainties about economics are rooted in our need for a better understanding of the economics of uncertainty. Such a better understanding of economic uncertainty is supplied by using Itô's methods.

More specifically, Merton [25] and [27] has argued that (1) is a satisfactory approximation of the actual behavior of certain economic variables. Furthermore, by using (1) to describe the stochastic behavior of economic variables and by assuming that trading takes place continuously in time, Merton states the benefits of such an analysis. The benefits include sharper results that are easier to interpret than those of the discrete time analysis and extensive mathematical literature on stochastic calculus which allows one to analyze rather complex economic models and still get quantitative results. In brief, stochastic models are considered because they lead to generalized results which are richer in theoretical content and more fruitful in empirical analysis.

*Received by the editors March 25, 1982, and in revised form January 12, 1983.
†Graduate School of Business, Loyola University of Chicago, Chicago, Illinois 60611.
In what follows we give the mathematical meaning of (1), we define the concepts of stochastic differential and stochastic differential equation, we state Itô's lemma and we give some mathematical examples. Then we proceed to formulate and solve two representative economic problems to illustrate the use of Itô's theory in modern financial decision making. We conclude with a few historical comments, bibliographical references and the basic remark that sometimes mathematical theories, no matter how pure or abstract, may eventually play a crucial part in applied problem solving.

2. Stochastic integration. Consider a probability space \((\Omega, \mathcal{F}, P)\) on which both a real-valued stochastic process \(X(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}\) and a real-valued Wiener process with unit variance \(Z(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}\) are defined. Equation (1) can be transformed into an integral equation to obtain

\[
X(t) - X(0) + \int_0^t f(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dZ(s).
\]

Note that \(X(0)\) is the initial condition which is a random variable. As a rule, the first integral in the right-hand side of (2) can be understood as a Riemann integral or in more general cases as a Lebesgue integral. However, the second integral presents a problem because \(dZ(t)\) does not exist. Putting it differently, although the Wiener process \(Z(t)\) is continuous with probability 1 (w.p.1), it is a function of unbounded variation w.p.1 and therefore the second integral cannot be interpreted as a Riemann–Stieltjes integral. Thus, unless we define the second integral in (2), the process \(X(t)\) has no meaning. The elements of stochastic integration are presented next to provide an appropriate meaning for the process \(X(t)\) in (2).

Let \(\{Z(t), t \geq 0\}\) be a Wiener process defined on a probability space \((\Omega, \mathcal{F}, P)\). A family of \(\sigma\)-fields \(\mathcal{F}_t\) in \(\mathcal{F}\) for \(t \geq 0\) is said to be nonanticipating with respect to \(Z(t)\) if it satisfies the following three conditions:

1. \(\mathcal{F}_t \subset \mathcal{F}_s\) for \(0 \leq s \leq t\).
2. \(\mathcal{F}_t\) contains the \(\sigma\)-field generated by \(Z(s)\) for \(0 \leq s \leq t\).
3. \(\mathcal{F}_t\) is independent of the \(\sigma\)-field generated by the increment \(Z(u) - Z(t)\), \(t \leq u < \infty\).

Condition 1 requires a monotonicity property to hold while condition 2 means that \(Z(t)\) is measurable with respect to \(\mathcal{F}_t\) for every \(t \geq 0\). Condition 3 means that for \(h = u - t, t \leq u < \infty\), the process \(Z(t + h) - Z(h)\) is independent of any of the events of the \(\sigma\)-field \(\mathcal{F}_h\). In particular, condition 3 means that \(\mathcal{F}_0\) can contain only events that are independent of the process \(Z(t)\) for \(t \geq 0\).

Consider now a function \(\sigma(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}\) which is assumed to be measurable in \((t, \omega)\) with \([0, T] \subset [0, \infty)\), and let \(\mathcal{F}_t, t \geq 0,\) be a nonanticipating family of \(\sigma\)-fields with respect to \(Z(t)\). A function \(\sigma(t, \omega)\) is said to be nonanticipating with respect to a family of \(\sigma\)-fields \(\mathcal{F}_t\) if it satisfies two conditions:

1. The sample path \(\sigma(t, \cdot)\) is \(\mathcal{F}_t\)-measurable for all \(t \in [0, T]\).
2. The integral

\[
\int_0^T |\sigma(t, \omega)|^2 \, dt
\]

is finite w.p.1.

Note that for a function \(\sigma(t, \omega)\) with continuous sample paths w.p.1, the last integral is an ordinary Riemann integral. In more general cases (3) is taken to be a Lebesgue integral.

A special class of nonanticipating functions is the class of nonanticipating step functions. A nonanticipating function \(\sigma(t, \omega)\) is called a step function if there exists a
partition of the interval \([0, T] \subset [0, \infty)\), say
\[
0 = t_0 < t_1 < \cdots < t_n = T < \infty,
\]
such that \(\sigma(t, \omega) = \sigma(t_i, \omega)\) for \(t \in [t_i, t_{i+1})\) for \(i = 0, 1, 2, \ldots, n - 1\). We remark that the points \(t_i\) are independent of \(\omega\). For such step functions we now define Itô’s stochastic integral.

Let \((\Omega, \mathcal{F}, P)\) be a probability space, \(\sigma(t, \omega) : [0, T] \times \Omega \to R\) a nonanticipating step function for a partition of the form of (4) and \(Z(t, \omega) : [0, T] \times \Omega \to R\) a Wiener process. The stochastic integral of a nonanticipating step function \(\sigma\) with respect to \(Z(t)\) over the interval \([0, T]\) is a real-valued random variable denoted by \(I(\sigma)\) and defined as
\[
I(\sigma) = \sum_{i=1}^{n} \sigma(t_{i-1}, \omega) [Z(t_i, \omega) - Z(t_{i-1}, \omega)]
\]
\[
= \sum_{i=1}^{n} \sigma(t_{i-1}) [Z(t_i) - Z(t_{i-1})] = \sum_{i=0}^{n-1} \sigma(t_i) [Z(t_{i+1}) - Z(t_i)].
\]
The presence of \(\omega \in \Omega\) emphasizes the fact that Itô’s integral is a random variable; \(\omega\) is sometimes omitted for notational convenience. It is important to remark that in the definition of the step function, and consequently in the definition of stochastic integral, the left-hand endpoints of the subintervals are used for evaluation. This definition is easily generalized to give Itô’s stochastic integral for an arbitrary nonanticipating function \(\sigma(t)\).

More specifically, let \((\Omega, \mathcal{F}, P)\) be a probability space, and consider a Wiener process \(Z(t)\) and an arbitrary nonanticipating function \(\sigma(t, \omega)\), both defined on \([0, T] \times \Omega\) and both real-valued. Itô’s stochastic integral of \(\sigma(t)\) with respect to \(Z(t)\) over the interval \([0, T]\), denoted by \(I(\sigma)\), is a random variable defined as the limit in probability (denoted \(\lim\) ) of the stochastic Cauchy sequence \(\{\int_{0}^{T} \sigma(t, \omega) dZ(t)\}\), that is,
\[
\int_{0}^{T} \sigma(t, \omega) dZ(t, \omega) \xrightarrow{P} \int_{0}^{T} \sigma(t, \omega) dZ(t, \omega) = I(\sigma).
\]
Here, \(\{\sigma_n(t)\}\) is a sequence of nonanticipating step functions that approximates \(\sigma\) in the sense of convergence in probability, that is,
\[
\int_{0}^{T} |\sigma(t, \omega) - \sigma_n(t, \omega)|^2 dt \xrightarrow{P} 0.
\]
We observe that \(I(\sigma)\) is unique w.p.1 and independent of the choice of the sequence \(\{\sigma_n(t)\}\) and we remark that Gikhman and Skorokhod [11, pp. 378–385] prove in detail, first, (7) and then (6).

Having defined Itô’s stochastic integral we next define the indefinite integral. Suppose that \(\chi_{[0,t]}\) is the characteristic function of the interval \([0, t] \subset [0, T]\) and let \(\sigma\) be a nonanticipating function on \([0, t]\) for each \(t\), where \(0 \leq t \leq T\), and where \(T\) is as before an arbitrarily large positive real number. The indefinite integral of a nonanticipating function \(\sigma(t)\) with respect to \(Z(t)\) is a stochastic process \(I(t)\) defined as follows:
\[
I(t) = \int_{0}^{T} \sigma(s, \omega) \chi_{[0,t]}(s) dZ(s, \omega).
\]
Note that \(I(t)\) is a real-valued \(\mathcal{F}_t\)-measurable stochastic process defined uniquely up to stochastic equivalence for \(t \in [0, T]\) with \(I(0) = \int_{0}^{T} \sigma(s) dZ(s) = 0\), w.p.1.

The above brief analysis clarifies the meaning of (2). Itô’s definite and indefinite integrals satisfy various important properties which are presented in Gikhman and Skorokhod [11, pp. 378–385]. For the purpose of our exposition these properties are not needed and therefore we proceed with additional concepts of Itô’s calculus.
3. Stochastic differentiation. In this section we define the concept of stochastic differential and we present the main result of stochastic differentiation which is the celebrated Itô's lemma. The various applications which follow demonstrate the immediate usefulness of these concepts.

Consider a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process $X(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$ that is measurable, for $t \in [0, T]$, with respect to a nonanticipating family of $\sigma$-fields $\mathcal{F}_t$, and a Wiener process $Z(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$. Assume that $\sigma(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a nonanticipating function on $[0, T]$ and also that $f(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}$ is measurable for each $t \in [0, T]$ with respect to $\mathcal{F}_t$ and also that $\int_0^T |f(t, \omega)| \, dt < \infty$, w.p.1. Let $0 \leq r < s \leq T$ and suppose

$$X(s) - X(r) = \int_r^s f(t, \omega) \, dt + \int_r^s \sigma(t, \omega) \, dZ(t, \omega).$$

The stochastic differential of the process $X(t)$ is defined to be the quantity $f(t) \, dt + \sigma(t) \, dZ(t)$ and is denoted as $dX(t)$, that is,

$$dX(t) = f(t) \, dt + \sigma(t) \, dZ(t).$$

A question naturally arises: Given a stochastic process $X(t)$ with respect to a Wiener process $Z(t)$ as in (9) if $Y(t) = u(t, X(t))$ be a new process, what is the stochastic differential of $Y(t)$ with respect to the same Wiener process? This question is important for both mathematical analysis and applications. The answer is provided below.

**Lemma 1** (Itô). Let $u(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous nonrandom function with continuous partial derivatives $u_t, u_x, \text{and } u_{xx}$. Suppose that $X(t) : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a process with stochastic differential

$$dX(t) = f(t) \, dt + \sigma(t) \, dZ(t).$$

Let $Y(t) = u(t, X(t))$. Then the process $Y(t)$ has a differential on $[0, T]$ given by

$$dY(t) = \left[ u_t(t, X(t)) + u_x(t, X(t)) f(t) + \frac{1}{2} u_{xx}(t, X(t)) \sigma^2(t) \right] \, dt$$

$$+ u_x(t, X(t)) \sigma(t) \, dZ(t).$$

The proof is presented in Gikhman and Skorokhod [11, pp. 387–389] and extensions of this lemma may be found in Arnold [1, pp. 90–99]. Here we limit our analysis by making three remarks.

First, Itô’s lemma is a useful result because it allows us to compute stochastic differentials of arbitrary functions having as an argument a stochastic process which itself is assumed to possess a stochastic differential. In this respect Itô’s formula is as useful as the chain rule of ordinary calculus.

Second, given an Itô stochastic process $X(t)$ with respect to a given Wiener process $Z(t)$ and letting $Y(t) = u(t, X(t))$ be a new process, Itô’s formula gives us the stochastic differential of $Y(t)$, where $dY(t)$ is given with respect to the same Wiener process.

Third, an inspection of the proof of Itô’s lemma reveals that it consists of an application of Taylor’s theorem of advanced calculus and several probabilistic arguments to establish the convergence of certain quantities to appropriate integrals. Therefore, the reader may obtain Itô’s formula by applying Taylor’s theorem instead of remembering the specific result in (11). More specifically, the differential of $Y(t) = u(t, X(t))$, where $X(t)$ is a stochastic process with differential given by (10), may be computed by using Taylor’s theorem and the following multiplication rules

$$dt \times dt = 0, \quad dZ \times dZ = dt, \quad dt \times dZ = 0.$$
as below:

\[
 dY(t) = u,dt + u_xdX(t) + \frac{1}{2}u_{xx}[dX(t)]^2
\]

\[
 = u,dt + u_x[f(t) dt + \sigma(t) dZ(t)]
+ \frac{1}{2}u_{xx}[f(t) dt + \sigma(t) dZ(t)]^2.
\]

Carrying out the multiplications in (13), using the rules in (12) and rearranging terms, we obtain Itô's results in (11).

We close this section by defining Itô's stochastic differential equation. Let us write the stochastic differential

\[
 dX(t) = f(t, X(t)) dt + \sigma(t, X(t)) dZ(t),
\]

which is exactly as (1) and suppose that the initial condition is given by

\[
 X(0, \omega) = X_0.
\]

An equation of the form (14) with initial condition as in (15) for \( t \in [0, T] \) is called an Itô stochastic differential equation. A stochastic process \( X(t) \) is called a solution of (14) given (15) on the interval \( [0, T] \) if \( X(t) \) satisfies the following three properties:

1. \( X(t) \) is \( \mathcal{F}_t \)-measurable, that is, nonanticipating for \( t \in [0, T] \).
2. Both integrals satisfy w.p.1

\[
 \int_0^T |f(t, X(t, \omega))| dt < \infty,
\]

\[
 \int_0^T |\sigma(t, X(t, \omega))|^2 dt < \infty.
\]

3. The integral equation

\[
 X(t) = X(0) + \int_0^t f(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dZ(s)
\]

is satisfied for every \( t \in [0, T] \) w.p.1.

4. Mathematical examples. Here we give two examples of a mathematical nature to illustrate the application of Itô's lemma before we present applications from financial economics.

**Example 1.** Suppose that \( Y(t) = u(X(t)) = e^{x(t)} \), where

\[
 dX(t) = -\frac{1}{2} dt + dZ(t)
\]

and \( X(0) = 0 \). Applying Itô's lemma, we obtain

\[
 dY(t) = u,dt + u_xdX(t) + \frac{1}{2}u_{xx}[dX(t)]^2
\]

\[
 = e^{x(t)}[-\frac{1}{2} dt + dZ(t)] + \frac{1}{2}e^{x(t)}[-\frac{1}{2} dt + dZ(t)]^2
\]

\[
 = e^{x(t)} dZ(t).
\]

This example illustrates that given the stochastic differential equation

\[
 dY(t) = e^{x(t)} dZ(t) - Y(t) dZ(t)
\]

with initial condition \( Y(0) = e^{x(0)} = 1 \), its solution is

\[
 Y(t) = \exp [-\frac{1}{2} t + Z(t)].
\]
This result illustrates the difference between ordinary differential equations and stochastic differential equations. Observe that if (16) were an ordinary differential equation its solution would be $Y(t) = \exp Z(t)$, which is different from (17).

Example 2. Suppose $Y(t) = u(X(t)) = u(Z(t))$, where $u$ is assumed to be twice continuously differentiable with respect to $x$; let $X(0) = Z(0) = 0$ and $dX(t) = dZ(t)$. Itô’s lemma gives

$$
dY(t) = u,dt + u, dX(t) + \frac{1}{2} u_{xx} [dX(t)]^2
\quad - u'(Z(t)) dZ(t) + \frac{1}{2} u''(Z(t)) dt.
$$

(18)

Here prime denotes ordinary derivative. If we write (18) in integral form, it becomes

$$
Y(t) = u(Z(t)) = u(0) + \int_0^t u'(Z(s)) dZ(s) + \frac{1}{2} \int_0^t u''(Z(s)) ds.
$$

(19)

From (19), solving for the second term in the right-hand side, we get

$$
\int_0^t u'(Z(s)) dZ(s) = u(Z(t)) - u(Z(0)) - \frac{1}{2} \int_0^t u''(Z(s)) ds.
$$

(20)

Equation (20) is called the fundamental theorem of calculus for Itô’s stochastic integral and it expresses the integral in the left-hand side only in terms of an ordinary integral.

We now proceed to illustrate the use of Itô’s calculus in financial economics.

5. The Pricing of an Option. An option is a contract giving the right to buy or sell an asset within a specified period of time subject to certain conditions. The simplest kind of an option is the European call option which is a contract to buy a share of a certain stock at a given date for a specified price. The date the option expires is called the expiration date or maturity date and the price that is paid for the stock when the option is exercised is called the exercise price or the striking price.

In terms of economic analysis several propositions about option pricing seem clear. The value of an option increases as the price of the stock increases. If the stock price is much greater than the exercise price, it is almost sure that the option will be exercised, and analogously, if the price of the stock is much less than the exercise price, the value of the option will be near zero and the option will expire without being exercised. If the expiration date is very far in the future, the value of the option will be approximately equal to the price of the stock. If the expiration date is very near, the value of the option will be approximately equal to the stock price minus the exercise price, or zero, if the stock price is less than the exercise price. In general the value of the option is more volatile than the price of the stock and the relative volatility of the option depends on both the stock price and maturity.

The first rigorous formulation and solution of the problem of option pricing was achieved by Black and Scholes [5] and Merton [24]. Consider a stock option denoted by $A$, whose price at time $t$ can be written as

$$
W(t) = F(t, S(t))
$$

(21)

where $F$ is a twice continuously differentiable function. Here $S(t)$ is the price of some other asset denoted by $B$, for example, the stock upon which the option is written. The price of $B$ is assumed to follow Itô’s stochastic differential equation,

$$
dS(t) = f(t, S(t)) dt + \eta(t, S(t)) dZ(t),
S(0) = S_0 \text{ given}.
$$
Assume as a simplifying special case that \( f(t, S(t)) = \alpha S(t) \) and that \( \eta(t, S(t)) = \sigma S(t) \) where \( \alpha \) and \( \sigma \) are constants. The last equation can be written as

\[
dS(t) = \alpha S(t) \, dt + \sigma S(t) \, dZ(t).
\]

Consider an investor who builds up a portfolio of three assets, \( A, B \) and a riskless asset, such as a government bond, denoted by \( C \). We assume that \( C \) earns the riskless competitive rate of return \( r(t) \). The nominal value of the portfolio is

\[
P(t) = N_1(t) S(t) + N_2(t) W(t) + Q(t),
\]

where \( N_1 \) denotes the number of shares of \( B \), \( N_2 \) the number of \( A \), and \( Q \) is the number of dollars invested in the riskless asset \( C \). Assume that \( B \) pays no dividends or other distributions. By Itô’s lemma using (21) and (22), we compute

\[
dW = F_t \, dt + F_S \, dS + \frac{1}{2} F_{SS} (dS)^2
\]

\[
= \left[ F_t + F_S \alpha S + \frac{1}{2} F_{SS} \sigma^2 S^2 \right] \, dt + F_S \sigma S \, dZ
\]

\[
= \alpha w \, W \, dt + \sigma w \, W \, dZ.
\]

Note that in (24) we let

\[
\alpha_w W = F_t + F_S \alpha S + \frac{1}{2} F_{SS} \sigma^2 S^2,
\]

\[
\sigma_w W = F_S \sigma S,
\]

in order to simplify the notation. The change in the nominal value of the portfolio, \( dP \), results from the change in the prices of the assets because at a point in time the quantities of option and stock are given, that is, \( dN_1 = dN_2 = 0 \). More precisely,

\[
dP = N_1(dS) + N_2(dW) + dQ
\]

\[
= (\alpha dt + \sigma dZ) N_1 S + (\alpha_w dt + \sigma_w dZ) N_2 W + r Q dt.
\]

Set \( w_1 = N_1 S / P, w_2 = N_2 W / P, w_3 = Q / P = 1 - w_1 - w_2 \). Then (25) becomes

\[
\frac{dP}{P} = (\alpha dt + \sigma dZ) w_1 + (\alpha_w dt + \sigma_w dZ) w_2 + (r dt) w_3.
\]

At this point the notion of economic equilibrium (also called risk-neutral or preference-free pricing) is introduced in the analysis. This notion plays an important role in modeling financial behavior and its appropriate formulation is considered to be a major breakthrough in financial analysis.

More specifically, design the proportions \( w_1, w_2 \) so that the position is riskless for all \( t \geq 0 \), that is, let \( w_1 \) and \( w_2 \) be such that

\[
\text{Var}_t \left( \frac{dP}{P} \right) - \text{Var}_t (w_1 \sigma dZ + w_2 \sigma_w dZ) = 0.
\]

In the last equation \( \text{Var} \) denotes variance conditioned on \( S(t), W(t) \) and \( Q(t) \). In other words, choose \( (w_1, w_2) = (\bar{w}_1, \bar{w}_2) \) so that

\[
\bar{w}_1 \sigma + \bar{w}_2 \sigma_w = 0.
\]

Then from (26), because the portfolio is riskless, it follows that

\[
E_t \left( \frac{dP}{P} \right) = [\alpha \bar{w}_1 + \alpha_w \bar{w}_2 + r(1 - \bar{w}_1 - \bar{w}_2)] \, dt = r(t) \, dt.
\]
Equations (28) and (29) yield the Black–Scholes–Merton equations

\[
\frac{\bar{w}_1}{\bar{w}_2} = -\frac{\sigma_w}{\sigma}
\]

and

\[
r = \alpha \bar{w}_1 + \alpha_w \bar{w}_2 - r \bar{w}_1 - r \bar{w}_2 + r,
\]

which simplify to

\[
\frac{\alpha - r}{\sigma} = \frac{\alpha_w - r}{\sigma_w}.
\]

Equation (32) says that the net rate of return per unit of risk must be the same for the two assets and describes an appropriate concept of economic equilibrium in this problem. Using (32) and making the necessary substitutions from (24a) and (24b), the partial differential equation of the pricing of an option is obtained.

\[
\frac{1}{2} \sigma^2 S^2 F_{ss}(t, S) + r S F_s(t, S) - r F(t, S) + F_c(t, S) - 0.
\]

6. A reexamination of option pricing. To illustrate the notion of economic equilibrium once again we present a modified exposition. Consider the nominal value of a portfolio consisting of a stock and a call option on this stock and write

\[
P(t) = N_1(t) S(t) + N_2(t) W(t)
\]

using the same notation as in section 5. Equation (34) differs from (23) by having deleted the term \(Q(t)\). Here we concentrate on the two assets of the portfolio, that is, the stock and call option. Using (34) and (24), the change in the value of the portfolio is given by

\[
dP = N_1 dS + N_2 dW - N_1 dS + N_2 [(F_i + \frac{1}{2} \sigma^2 S^2) dt + F_s dS].
\]

Note that \(dN_1 - dN_2 = 0\), since at any given point in time the quantities of stock and option are given. For arbitrary quantities of stock and option, (35) shows that the change in the nominal value of the portfolio, \(dP\), is stochastic because \(dS\) is a random variable expressed in (22). Suppose the quantities of stock and call option are chosen so that

\[
\frac{N_1}{N_2} = -F_s.
\]

Then \(N_1 dS + N_2 F_s dS = 0\), and inserting (36) into (35) yields

\[
dP = -N_2 F_s dS + N_2 [(F_i + \frac{1}{2} \sigma^2 S^2) dt + F_s dS] = N_2(F_i + \frac{1}{2} \sigma^2 S^2) dt.
\]

Let \(N_2 = 1\) in (37) and observe that in equilibrium the rate of return of the riskless portfolio must be the same as the riskless rate \(r(t)\). Therefore we write

\[
\frac{dP}{P} = r dt.
\]

Equation (38) can be used to derive the partial differential equation for the value of the option. Making the necessary substitutions in (38), obtain

\[
\frac{(F_i + (1/2) \sigma^2 S^2) dt}{-F_s S + W} = r dt,
\]

which upon rearrangement gives (33). Note that the option pricing equation in (33) is a second-order linear partial differential equation of the parabolic type. The boundary
conditions of (33) are determined by the specification of the asset. For the case of an option which can be exercised only at the expiration date $t^*$ with an exercise price $E$, the boundary conditions are

(40a) $F(t, S = 0) = 0$,

(40b) $F(t = t^*, S) = \max \{0, S - E\}$.

Observe that (40a) says that the call option price is zero if the stock price is zero at any date $t$; (40b) says that the call option price at the expiration date $t = t^*$ must equal the maximum of either zero or the difference between the stock price and the exercise price.

The solution of the option pricing equation subject to the boundary conditions is given by Black and Scholes [5] and Merton [24] as:

(41) $F(T, S, \sigma^2, E, r) = S\Phi(d_1) - e^{rT} \Phi(d_2)$,

where $\Phi$ denotes the cumulative normal distribution, namely,

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} dx.$$ 

In (41), $T = t^* - t$, namely, $T$ is time to expiration (measured in years) and $d_1$ and $d_2$ are given by

(42) $d_1 = \ln(S/E) + (r + \sigma^2/2) T / \sigma \sqrt{T}$,

(43) $d_2 = \ln(S/E) + (r - \sigma^2/2) T / \sigma \sqrt{T}$.

It can be shown that

$$\frac{\partial F}{\partial T} > 0, \quad \frac{\partial F}{\partial S} > 0, \quad \frac{\partial F}{\partial \sigma^2} > 0, \quad \frac{\partial F}{\partial E} < 0, \quad \frac{\partial F}{\partial r} > 0.$$ 

These partial derivatives justify the intuitive behavior of the price of an option as was indicated in the beginning of 5. More specifically, these partials show that as the stock price rises so does the option price; as the variance rate of the underlying stock rises so does the option price; with a higher exercise price, the expected payoff decreases; the value of the option increases as the interest rate rises; and finally, with a longer time to maturity the price of the option is greater.

7. An example. Equation (41) indicates that the Black–Scholes option pricing model is a function of only five variables: $T$, the time to expiration; $S$, the stock price; $\sigma^2$, the instantaneous variance rate on the stock price; $E$, the exercise price; and $r$, the riskless interest rate. From these five variables, only the variance rate must be estimated; the other four variables are directly observable. A simple example is presented to illustrate the use of (41). The values of the observable variables are taken from the Wall Street Journal.

On Friday, December 10, 1982, the IBM stock price was $92.875. To estimate the call option price with expiration date the third Friday in April 1983, with an exercise price $95, the riskless rate and the instantaneous variance need to be estimated. The riskless rate is estimated by using the average of the bid and ask quotes on U.S. Treasury bills of
approximately the same maturity as the option. The results of the Monday, December 13, 1982, auction show a riskless rate of 7.995% for U.S. Treasury bills maturing in 13 weeks. The only missing piece of information is the instantaneous variance of the stock price.

There are several different techniques which have been suggested for estimating the instantaneous variance. In this regard the work of Latané and Rendleman [20] must be mentioned; they derive standard deviations of continuous price relative returns which are implied in actual call option prices on the assumption that investors behave as if they price options according to the Black–Scholes model. In our example we calculate the implicit variance by using the actual April 1983 call price of an IBM option with an exercise price of $90 to solve for an estimate of the instantaneous variance. More specifically, a numerical search is used to approximate the standard deviation implied by the Black–Scholes formula with parameters: price of the stock, $S = 92.875$; exercise price, $E = 90$; time to expiration, $T = 126/365 = .345$; riskless rate, $r = .08$; and call option price, $F = 9.875$. The approximated implicit standard deviation is found to be $\sigma = .35$.

After the above clarifications are made, the example is this: given $S = 92.875$, $E = 95$, $T = .345$, $r = .08$ and $\sigma = .35$, use (41) to compute $F$. Using (42) and (43) we calculate

\[
\begin{align*}
d_1 &= \frac{\ln (92.875/95) + [.08 + (1/2)(.35)^2](.345)}{.35 \sqrt{.345}} = .127004, \\
d_2 &= \frac{\ln (92.875/95) + [.08 - (1/2)(.35)^2](.345)}{.35 \sqrt{.345}} = -.078575.
\end{align*}
\]

From statistical tables, giving the area of a standard normal distribution, we obtain $\Phi(.127004) = .550532$ and $\Phi(-.078575) = .46867$. Finally,

\[
F = (92.875)(.550532) - \frac{95}{e^{(1/2)(.345)^2}(46867)} = 7.819.
\]

The calculated call option price of $7.819 is very close to the actual call price of $7.875 reported in the Wall Street Journal on Monday December 13, 1982.

This simple example shows how to use the Black–Scholes model to price a call option under the assumptions of the model. The example is presented for illustrative purposes only and it relies heavily on the implicit estimate of the variance, its constancy over time and all the remaining assumptions of the model. The appropriateness of estimating the instantaneous variance implicitly is ultimately an empirical question, as is the entire Black–Scholes pricing formula. Boyle and Ananthanarayanan [6] study the implications of using an estimate of the variance in option valuation models and show that this procedure produces biased option values. However, the magnitude of this bias is not large.

One additional remark must be made. The closeness in this example of the calculated call option price to the actual call price is not necessarily evidence of the validity of the Black–Scholes model. Extensive empirical work has taken place to investigate how market prices of call options compare with prices predicted by Black and Scholes. The interested reader is referred to Macbeth and Merville [21] and Bhattacharya [4].

8. Remarks on option pricing. It is beyond the scope of this paper to review the voluminous literature on option pricing. For such a review the reader is referred to the two papers by Smith [31] and [32]. It is appropriate, however, to make a few remarks on the Black–Scholes option pricing model to clarify its significance and its limitations.

First, the Black–Scholes model for a European call as derived in [5] and [24] and as
reported here is based on several simplifying assumptions: the stock price follows an Itô equation; the market operates continuously; there are no transaction costs in buying or selling the option or the underlying stock; there are no taxes; the riskless rate is known and constant; and finally, there are no restrictions on short sales. Several researchers have extended the original Black-Scholes model by modifying these assumptions. Merton [24] generalizes the model to include dividend payments, exercise price changes and the case of a stochastic interest rate. Roll [29] has solved the problem of valuing a call option that can be exercised prior to its expiration date when the underlying stock is assumed to make known dividend payments before the option matures. Ingersoll [13] studies the effect of differential taxes on capital gains and income while Scholes [30] determines the effects of the tax treatment of options on the pricing model. Furthermore, Merton [26] and Cox and Ross [8] have shown that if the stock price movements are discontinuous, under certain assumptions, the valuation model still holds. These, and some other modifications of the original Black–Scholes analysis, have shown the model to be quite robust regarding the relaxation of its foundational assumptions.

Second, it is worth repeating that the use of Itô’s calculus and the important insight concerning the appropriate concept of an equilibrium by creating a riskless hedge portfolio have led Black and Scholes to obtain a closed form solution for option pricing. In this closed form solution several variables do not appear, such as: the expected rate of return on the stock, the expected rate of return on the option, a measure of investor’s risk preference, investor expectations and equilibrium conditions for the entire capital market.

Third, the Black–Scholes pricing model has found numerous applications. We mention a few such as: pricing the debt and equity of a firm; the effects of corporate policy and, specifically, the effect of mergers, acquisitions and scale expansions on the relative values of the debt and equity of the firm; the pricing of convertible bonds; the pricing of underwriting contracts; the pricing of leases; and the pricing of insurance. Smith in [31] and [32] summarizes most applications and indicates the original references.

Finally, three important papers by Harrison and Kreps [12], Kreps [17] and [18] consider some foundational issues that arise in conjunction with the arbitrage theory of option pricing. The important point to consider is this: the ability to trade securities frequently can enable a few multiperiod securities to span many states of nature. In the Black–Scholes theory there are two securities and uncountably many states of nature, but because there are infinitely many trading opportunities and because uncertainty resolves nicely, markets are effectively complete. Thus, even though there are far fewer securities than states of nature, nonetheless, markets are complete and risk is allocated efficiently.

9. Term structure of interest rates. The term structure of interest rates measures the relationship among the yields on default free securities that differ only in their term to maturity. The determinants of this relationship have long been an area of active research for economists. By offering a complete schedule of interest rates across time, the term structure embodies the market’s anticipations of future events. Therefore, an explanation of the term structure gives researchers a way to extract this information and encourages them to develop testable theories.

Previous theories of the term structure have assumed as their starting point a world of certainty and have proceeded by examining stochastic generalizations of the certainty equilibrium relationships. The literature in this area is voluminous and a reexamination of several traditional theories, while employing recent advances in the theory of valuation of contingent claims, can be found in Cox, Ingersoll and Ross [7]. Here, we present a term structure model developed by Vasicek [34].
Consider a market in which investors issue and buy default free claims on a specified sum of money to be delivered at a given future date. Such a claim will be called a **discount bond**.

Let \( P(t, s) \) denote the price at time \( t \) of a discount bond maturing at time \( s \), with \( t \leq s \). The bond is assumed to have a maturity value of one unit, that is, \[
(44) \quad P(s, s) = 1.
\]

The **yield to maturity** denoted by \( R(t, T) \) is the internal rate of return at time \( t \) on a bond with maturity date \( s = t + T \), and is given by

\[
(45) \quad R(t, T) = -\frac{1}{T} \ln P(t, t + T), \quad T > 0.
\]

From (45), the rates \( R(t, T) \) considered as a function of \( T \) will be referred to as the **term structure at time \( t \)**. We use (45) to define the **spot rate** as the instantaneous borrowing and lending rate, \( r(t) \), given by

\[
(46) \quad r(t) = R(t, 0) - \lim_{T \to 0} R(t, T).
\]

At any time \( t \), the current value \( r(t) \) of the spot rate is known. However, the subsequent future values are not necessarily certain. It is natural to expect that the price of a discount bond will be determined over its term solely by the spot interest rate. More accurately, we assume that the price \( P(t, s) \) of the discount bond is determined by the assessment at time \( t \) of the spot rate process \( \{r(u), t \leq u \leq s\} \) over the term of the bond. We write

\[
(47) \quad P(t, s) = P(t, s, r(t))
\]

to indicate that the price \( P(t, s) \) of the discount bond is a function of the spot rate \( r(t) \). To complete the model we need to postulate the behavior of the spot rate process. It is assumed that the stochastic dynamics of the spot rate can be described by an Itô stochastic differential equation given by

\[
(48) \quad dr(t) = f(t, r(t)) \, dt + \sigma(t, r(t)) \, dZ(t).
\]

Finally, assume that there are no transaction costs, information is available to all investors simultaneously and that investors act rationally, that is, assume that the market is efficient. This last assumption implies that no profitable riskless arbitrage is possible.

From (47) and (48) by using Itô's lemma obtain the stochastic differential equation

\[
(49) \quad dP = P \mu(t, s, r(t)) \, dt - P \sigma(t, s, r(t)) \, dZ
\]

which describes the bond price changes. In (49) the functions \( \mu \) and \( \sigma \) are defined as follows,

\[
(50) \quad \mu(t, s, r) = \frac{1}{P(t, s, r)} \left[ \frac{\partial}{\partial t} + f \frac{\partial}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial r^2} \right] P(t, s, r),
\]

\[
(51) \quad \sigma(t, s, r) = -\frac{1}{P(t, s, r)} \rho \frac{\partial}{\partial r} P(t, s, r).
\]

Consider now the quantity \( q(t, r(t)) \) given by

\[
(52) \quad q(t, r) = \frac{\mu(t, s, r) - r}{\sigma(t, s, r)}, \quad t \leq s,
\]

which is called the **market price of risk** and which specifies the increase in expected instantaneous rate of return on a bond per an additional unit of risk. Substitute the
expressions of \( \mu \) and \( \sigma \) from (50) and (51) into (52), make the necessary rearrangements to obtain the term structure equation given by

\[
\frac{\partial P}{\partial t} + (f + \rho q) \frac{\partial P}{\partial r} + \frac{1}{2} \rho^2 \frac{\partial^2 P}{\partial r^2} - rP = 0.
\]

Observe that (53) is a partial differential equation whose solution \( P \) may be obtained once the spot rate process \( r(t) \) and market price of risk \( q(t, r) \) are specified. The boundary condition of (53) is

\[
P(s, s, r) = 1.
\]

Knowing \( P(t, s, r) \) as a solution of (53) subject to (54) allows one to obtain the term structure from (45).

Vasicek uses techniques presented in Friedman [10] to write a representation for the bond price as a solution to the term structure equation, given by

\[
P(t, s) = E_t \exp \left[ -\int_t^s r(u) \, du - \frac{1}{2} \int_t^s q^2(u, r(u)) \, du + \int_t^s q(u, r(u)) \, dZ(u) \right],
\]

\( t \leq s \).

To obtain some economic insight in (55), construct a portfolio consisting of a bond whose maturity approaches infinity, called a long bond, and lending or borrowing at the spot rate, with proportions \( \lambda(t) \) and \( 1 - \lambda(t) \) respectively. Define \( \lambda(t) \) as

\[
\lambda(t) = \frac{\mu(t, \infty) - r(t)}{\sigma^2(t, \infty)},
\]

that is,

\[
\lambda(t) \sigma(t, \infty) = q(t, r(t)).
\]

The price, \( Q(t) \), of such a portfolio satisfies the equation

\[
dQ - \lambda Q[\mu(t, \infty) \, dt - \sigma(t, \infty) \, dZ] + (1 - \lambda) Qr \, dt.
\]

Equation (57) can be integrated by evaluating the differential of \( \ln Q \) and using (56) to yield

\[
d(\ln Q) = \lambda \mu(t, \infty) \, dt - \lambda \sigma(t, \infty) \, dZ + (1 - \lambda) \, r \, dt - \frac{1}{2} \lambda \sigma^2(t, \infty) \, dt
\]

\[= rd\tau + \frac{1}{2} q^2 \, dt - qdZ.
\]

Therefore, we conclude that

\[
\frac{Q(t)}{Q(s)} = \exp \left[ -\int_t^s r(u) \, du - \frac{1}{2} \int_t^s q^2(u, r(u)) \, du + \int_t^s q(u, r(u)) \, dZ \right].
\]

Using this last equation we can rewrite (55) as

\[
P(t, s) = E_t \frac{Q(t)}{Q(s)}, \quad t \leq s.
\]

This means that the price of any bond measured in units of the value of a portfolio \( Q \) follows a martingale,

\[
\frac{P(t, s)}{Q(t)} = E_t \frac{P(u, s)}{Q(u)}, \quad t \leq u \leq s.
\]
Therefore, we conclude that if the bond price at time \( t \) is a certain fraction of the value of the portfolio \( Q \), then the same will hold in the future. Further applications of the martingale concept in financial economics are presented in Malliaris [22] and an arbitrage model of the term structure of interest rates may be found in Richard [28].

10. Concluding remarks. In this final section we collect various remarks, historical notes and additional bibliographical references.

Stochastic integration was developed by Itô [14] as he generalized a stochastic integral first introduced in 1923 by Wiener [35]. Parts of Itô’s original work were presented initially by Doob [9] and later by Gikhman and Skorokhod [11]. Our presentation on stochastic integration follows Gikhman and Skorokhod [11, pp. 378–385], Bharucha-Reid [3, pp. 221–226], Arnold [1, pp. 64–75], Friedman [10, pp. 55–72] and Ladde and Lakshmikantham [19, pp. 114–122].

It is useful to remark that Itô’s stochastic integral is not related to the various nonstochastic integrals of ordinary calculus and measure theory. Doob [9, p. 444] shows that if \( Z(t, \omega) \) is a Wiener process with unit variance, then Itô’s stochastic integral yields

\[
\int_a^b Z(t, \omega) \, dZ(t, \omega) = \frac{1}{2} [Z^2(b, \omega) - Z^2(a, \omega)] - \frac{1}{2} (b - a),
\]

which is different from the ordinary Riemann integral for a continuous nonstochastic function \( Z(t) \), which is

\[
\int_a^b Z(t) \, dZ(t) - \frac{1}{2} [Z^2(b) - Z^2(a)].
\]

Since Itô’s calculus is based on Itô’s stochastic integral, it is not surprising that Itô’s rules of stochastic differentiation are not the same as the rules of ordinary calculus. Furthermore, Itô’s stochastic differential and integral equations differ from the ordinary ones. Therefore, it may be concluded that Itô’s calculus is an independent, self-consistent mathematical theory which is not connected to ordinary calculus. In other words, Itô’s calculus cannot be formulated as an extension, in some mathematical sense, of ordinary calculus.

Stratonovich [33] has defined a new stochastic integral as a mean square limit of a symmetrized sum which preserves the rules of ordinary calculus. However, the economics and finance literature relies almost exclusively on Itô’s stochastic integral and Itô’s calculus in general for several reasons. The expectations of the Itô integral are easier to compute than the Stratonovich integral; the Itô has the important property of being a martingale; the Itô stochastic differential equation exhibits a nonanticipating property which is the appropriate way to model economic uncertainty; the solutions of Itô equations have nice properties, that is, they are Markov process and under certain conditions they are diffusion processes; and finally, virtually all the mathematical theory of stochastic control and stability is developed using the Itô integral.

Itô’s lemma first appeared in Itô [15] and later in Itô [16]. In our presentation we follow Gikhman and Skorokhod [11, pp. 387–389] and Arnold [1, pp. 96–99]. The mathematical examples and financial applications presented in this paper illustrate the usefulness of Itô’s lemma. Note also that Itô’s lemma has found further applications in financial economics in areas such as: first, the formulation of appropriate stochastic budget constraints and, second, the stochastic control problems of optimum consumption and portfolio decisions. A detailed presentation of these topics may be found in Malliaris and Brock [23].
Acknowledgments. The ideas discussed in this paper were presented in some detail at a Finance Seminar at The University of Texas at Austin in June 1981. I am thankful to all members of this seminar for their interest and in particular to Professors Stephen Magee, Sam Cox, Wayne Lee, George Morgan, Stephen Smith and Robert Witt for helpful discussions. I am also thankful to Carol Ross for research and computational assistance. A briefer version of this paper was presented at the Midwest Conference of the American Institute for Decision Sciences, April 7–9, 1982, Milwaukee, Wisconsin.

REFERENCES


